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Periodic structure of the effective fields for a regular Ising model on the Cayley tree at $T = 0$

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Abstract. The regular Ising model with l bonds of $J > 0$ and l bonds of $-J$ on the Cayley tree with coordination number $2l$ is considered under the uniform external field when $l \geq 3$. The effective fields in the system satisfy nonlinear coupled recursive equations. At $T = 0$, these become piecewise linear equations and arithmetic furcations of periodic points for the recursive equations are found as a function of the external field.

The thermodynamic behaviour of the Ising model at the central part of the Cayley tree is the same as the one of the Ising model on the Bethe lattice (Morita and Horiguchi 1975). The Bethe lattice is defined by the lattice on which the Bethe approximation is exact for the Ising model. The regular Ising model with ferromagnetic nearest-neighbour (NN) and antiferromagnetic next-nearest-neighbour (NNN) interactions on the Cayley tree with coordination number three has been studied, paying special attention to the properties at the central site of the tree (Vannimenus 1981, Inawashiro and Thompson 1982). Vannimenus (1981) has obtained the modulated phase for the model in which the NNN interactions between sites on the same shell are vacant. Inawashiro and Thompson (1982) have obtained a chaotic, glass-like behaviour for the model in which all of the NNN interactions participate. In the models by Vannimenus and by Inawashiro and Thompson, there exists frustration due to the competing NN and NNN interactions. In this sense, the situation is the same as the ANNNI model (Selke and Fisher 1979, Fisher and Selke 1980, Bak and von Boehm 1980, Yokoi *et al* 1981).

On the other hand, Morita (1983) has studied a regular Ising model with only nearest-neighbour interactions under a uniform external field on the Cayley tree with the coordination number three. In the model, there are two $J > 0$ bonds and one $-J$ bond from each site to its nearest-neighbour sites except those on the surface. He obtained the spin-glass and the spin-crystal phases in the temperature–field plane. In the model, frustration exists due to the competition between the exchange interaction and the applied external field. This is the point essentially different from the models studied by others. In the present paper, we investigate the ground-state properties of a model modified from it, focusing on the furcation of the attractor of the recursive equations for the effective fields.

We consider the Cayley tree which has a central site 0 and N shells surrounding it. The coordination number is denoted by z, which is equal to or greater than three. We label the shells in order from the outermost to the innermost; then the central

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site is on the Nth shell. The sites other than those on the 0th shell (referred to as the surface of the Cayley tree) have z nearest neighbours and those on the surface have only one nearest neighbour. There is a spin on each site. Each spin, except on the surface, interacts with l_+ nearest-neighbour spins by $J>0$ and l_- nearest-neighbour spins by $-J$, where $l_+ + l_- = z \geqslant 3$. We introduce two types of effective fields $h_s^{(+)}$ or $h_s^{(-)}$ to a site on the sth shell from the outer branch, according as the interaction between them is J or $-J$. The equations determining the effective fields $h_s^{(\pm)}$ except for $s=0$ are given as follows:

$$\begin{aligned} h_s^{(+)} &= \beta^{-1}\tanh^{-1}(\tanh\beta J \tanh\{\beta[h+(l_+-1)h_{s-1}^{(+)}+l_-h_{s-1}^{(-)}]\}), \\ h_s^{(-)} &= -\beta^{-1}\tanh^{-1}(\tanh\beta J \tanh\{\beta[h+l_+h_{s-1}^{(+)}+(l_--1)h_{s-1}^{(-)}]\}), \end{aligned} \tag{1}$$

where $\beta = 1/kT$ as usual. h is the uniformly applied external field. At the surface $s=0$, we assume $h_0^{(\pm)}=0$. This set of equations is a set of nonlinear coupled difference equations. In the temperature–field plane, there are regions of periodic points, of limit periodic points and of non-periodic points for sets of values $(h_s^{(+)}, h_s^{(-)})$ of the recurrence equations (1). At a point in a region of non-periodic points, we have an attractor which is a closed curve. Correlation functions of the effective fields seem to be almost periodic on the attractor. Detailed discussions for finite temperatures will be given elsewhere.

In the present paper, we concentrate our attention on the properties at absolute zero temperature. Furthermore, we restrict the number of J bonds to be equal to that of $-J$ bonds and greater than or equal to three: $l \equiv l_+ = l_- \geqslant 3$. At $T=0$, equation (1) is simply expressed by the mapping Φ from R^2 into the set $\{x, y \,||x| \leqslant J, |y| \leqslant J\}$:

$$h_s^{(\pm)} = \tfrac{1}{2}\{|h+(l-1)h_{s-1}^{(\pm)}+lh_{s-1}^{(\mp)} \pm J| - |h+(l-1)h_{s-1}^{(\pm)}+lh_{s-1}^{(\mp)} \mp J|\}. \tag{2}$$

We denote this as follows

$$(h_s^{(+)}, h_s^{(-)}) = \Phi(h_{s-1}^{(+)}, h_{s-1}^{(-)}) = \Phi^k(h_{s-k}^{(+)}, h_{s-k}^{(-)}). \tag{3}$$

Here Φ is continuous and piecewise linear. When $h<2J$, we have only an unstable fixed point $(0, -h/l)$. When $h \geqslant 2J$, we have a stable fixed point $(J, -J)$, and an additional unstable fixed point $(0, -h/l)$ if $lJ \geqslant h \geqslant 2J$. (When $l=2$ and $h=2J$, we also have a line of stable fixed points which is expressed by the set $\{x, y\,|0 \leqslant x \leqslant J, y=-J\}$.) When $h \geqslant 2J$, every point in R^2 except unstable fixed points is attracted to the stable fixed point $(J, -J)$ by the mapping Φ. Hence we investigate in detail only the case of $h<2J$.

We look at the following domains:

$$E = \{x, y\,||h+(l-1)x+ly|<J, |h+lx+(l-1)y|<J\}, \tag{4}$$

$$E_5 = \{x, y\,|(x, y)\in E, |x|<J/(2l-1), |y+2h/(2l-1)|<J/(2l-1)\}, \tag{5}$$

$$F = \{x, y\,||h+(l-1)x+ly|<J, h+lx+(l-1)y \geqslant J\}. \tag{6}$$

It can easily be seen that every point (x, y) in $R^2\backslash E_5$ is mapped to a point in F at most within five operations of the mapping. Every point in E_5 is mapped to a point of E and then possibly to a point in E_5 itself. When the points $(h_{s+j}^{(+)}, h_{s+j}^{(-)})$ are in E_5

for $j = 0, 1, 2, \ldots, 2n-1$, we have the following relations:

$$\begin{aligned} h^{(+)}_{s+2n-1} &= (1-2l)^{n-1}\{(l-1)h^{(+)}_s + l(h^{(-)}_s + h/l)\}, \\ h^{(-)}_{s+2n-1} &= -(1-2l)^{n-1}\{lh^{(+)}_s + (l-1)(h^{(-)}_s + h/l)\} - h/l, \end{aligned} \tag{7}$$

$$\begin{aligned} h^{(+)}_{s+2n} &= (1-2l)^n h^{(+)}_s, \\ h^{(-)}_{s+2n} &= (1-2l)^n (h^{(-)}_s + h/l) - h/l. \end{aligned} \tag{8}$$

Thus after $2N$ iterations, any point $(h^{(+)}_s, h^{(-)}_s)$ in E_5 which satisfies $|h^{(+)}_s| > J/(2l-1)^{N+1}$ and $|h^{(-)}_s + h/l| > (J + h/l)/(2l-1)^{N+1}$ is mapped to a point outside E_5. In this way,

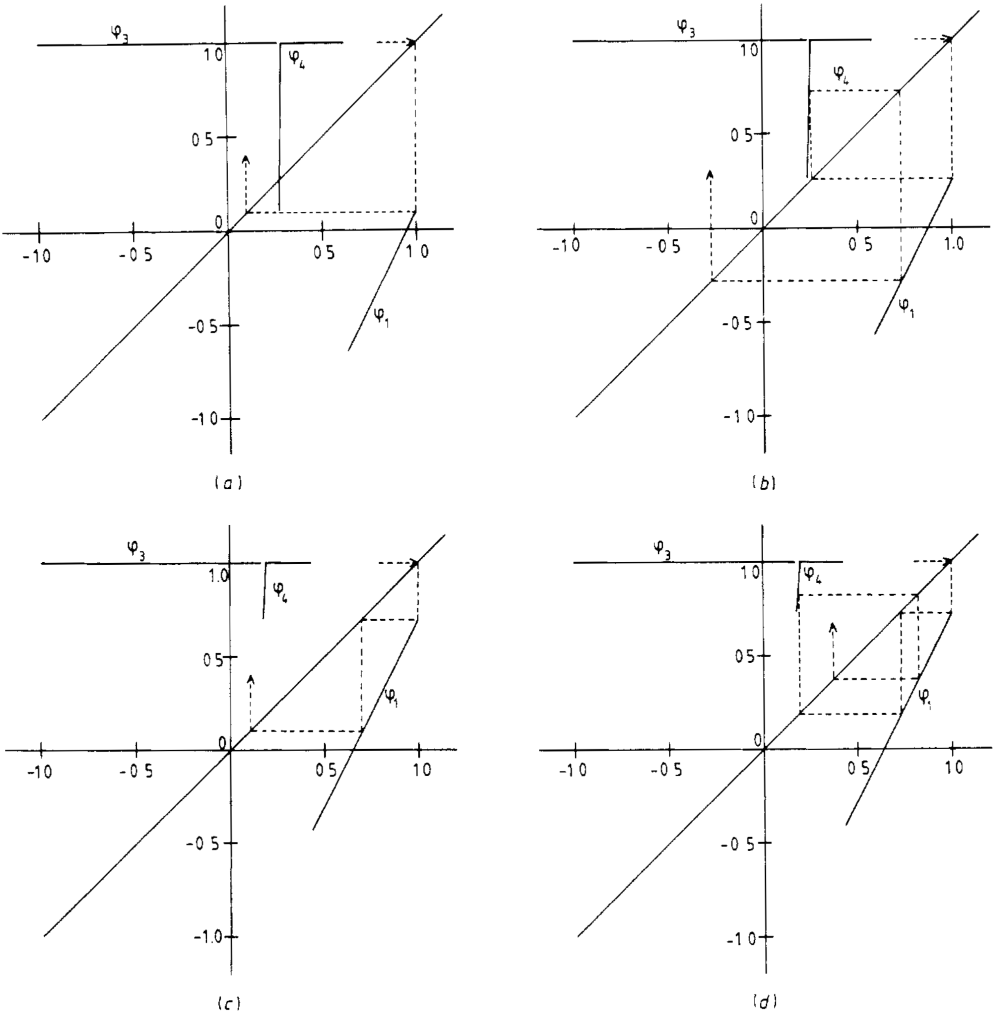

Figure 1. The one-dimensional mapping φ_k and the processes of furcations of periodic points. The ordinate is $\varphi_k(x)/J$ and the abscissa is x/J. (a) is for $h/J = 1.1$, (b) for $h/J = 1.255$, (c) for $h/J = 1.7$ and (d) for $h/J = 1.728$. From (a) to (c), the furcation from period 4 to period 5 is realised. From (a) to (b), the furcation to period 9 in between periods 4 and 5 is realised. From (c) to (d), the furcation to period 11 in between periods 5 and 6 is realised.

every point except the unstable fixed point $(0, -h/l)$ passes through a point in F and then is mapped to a point in the set $\{x, y \| x| \leqslant J, y = -J\}$.

The above fact suggests that we should consider a projection of the two-dimensional mapping Φ to the one-dimensional mapping φ_k defined by

$$h^{(+)}_{s+k} = \varphi_k(h^{(+)}_s). \tag{9}$$

Here equation (9) denotes the following equation:

$$(h^{(+)}_{s+k}, h^{(-)}_{s+k} = -J) = \Phi^k(h^{(+)}_s, h^{(-)}_s = -J) \tag{10}$$

where for $k \neq 1$ we assume that $h^{(-)}_{s+m} \neq -J$ for $1 \leqslant m < k$. In equation (9), the subscript k of φ_k is also a function of $h^{(+)}_s$. For $l = 3$ it is expressed as follows: for $h/J \leqslant \frac{28}{17}$

$$\begin{aligned}
\varphi_1(x) &= -3J + h + 2x, & J - \tfrac{1}{3}h &< x \leqslant J, \\
\varphi_4(x) &= J, & \tfrac{41}{81}J - \tfrac{16}{81}h &< x \leqslant J - \tfrac{1}{3}h, \\
\varphi_4(x) &= -40J + 16h + 81x, & \tfrac{13}{27}J - \tfrac{5}{27}h &< x \leqslant \tfrac{41}{81}J - \tfrac{16}{81}h, \\
\varphi_3(x) &= J, & -J \leqslant x &\leqslant \tfrac{13}{27}J - \tfrac{5}{27}h,
\end{aligned} \tag{11}$$

and for $h/J \geqslant \frac{28}{17}$

$$\begin{aligned}
\varphi_1(x) &= -3J + h + 2x, & J - \tfrac{1}{3}h &< x \leqslant J, \\
\varphi_4(x) &= J, & \tfrac{1}{9}J + \tfrac{2}{45}h &< x \leqslant J - \tfrac{1}{3}h, \\
\varphi_4(x) &= -4J - 2h + 45x, & \tfrac{1}{15}J + \tfrac{1}{15}h &< x \leqslant \tfrac{1}{9}J + \tfrac{2}{45}h, \\
\varphi_3(x) &= J, & -J \leqslant x &\leqslant \tfrac{1}{15}J + \tfrac{1}{15}h.
\end{aligned} \tag{12}$$

The function φ_k is also piecewise linear. φ_k is given in figure 1 by bold full lines labelled by φ_1, φ_3 and φ_4 for $h/J = 1.1$, 1.255, 1.7 and 1.728. The processes of the mapping are shown by the broken lines. Suppose that the broken line starting from the point (J, J) visits φ_1, φ_3 and φ_4 at ν_1, ν_3 and ν_4 times, respectively, before it comes back to the same point; then the period is given by $\nu_1 + 3\nu_3 + 4\nu_4$. We have period 4, 9, 5 and 11 in figures 1(a), (b), (c) and (d), respectively. The arithmetic furcations of periods are explained in the following way. The series of periods $j = 4$,

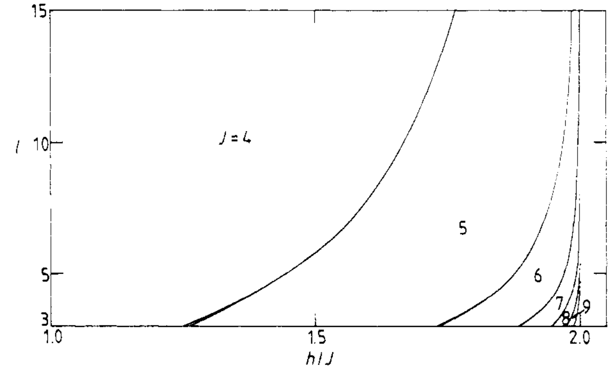

Figure 2. The regions for the main series of periods $j = 4, 5, \ldots$ in the h–l plane for $l \geqslant 3$. In between periods j and $j+1$, there exist regions with periods $jm + (j+1)n$ for $m, n \in \{1, 2, \ldots\}$.

5, 6, ... is called the main series and is obtained by increasing the number of times of visiting φ_1 before φ_3 or φ_4 or by switching from φ_3 to φ_4 after φ_1, and the periods in between successive main periods are obtained by visiting the tongue part of φ_4.

We performed numerical calculations and actually found arithmetic furcations of the periodic points as a function of the external field. The periods $jm+(j+1)n$ appear in decreasing order of the value $(m+n)/[jm+(j+1)n]$, where m and n take on {0, 1,

Table 1. The critical values of h at which the period changes are listed within ten places of decimals for those occurring in between period 4 and period 10.

Field h/J	Period	$(m+n)/[jm+(j+1)n]$
0–1.25	4	0.25
1.250 000 0001	25	0.24
1.250 000 0002–1.250 000 0145	21	0.238
1.250 000 0146–1.250 001 1760	17	0.235
1.250 001 1761–1.250 095 2551	13	0.231
1.250 095 2552–1.250 095 2622	22	0.227
1.250 095 2623–1.257 685 0094	9	0.222
1.257 685 0095–1.257 685 0130	23	0.217
1.257 685 0131–1.257 731 6689	14	0.214
1.257 731 6690–1.257 731 9569	19	0.211
1.257 731 9570–1.257 731 9587	24	0.208
1.257 731 9588–1.727 272 7272	5	0.2
1.727 272 7273	31	0.194
1.727 272 7274–1.727 272 7300	26	0.192
1.727 272 7301–1.727 272 9795	21	0.190
1.727 272 9796–1.727 295 4285	16	0.188
1.727 295 4286–1.727 295 4299	27	0.185
1.727 295 4300–1.729 312 0638	11	0.182
1.729 312 0639–1.729 312 0644	28	0.179
1.729 312 0645–1.729 323 2458	17	0.176
1.729 323 2459–1.729 323 3079	23	0.174
1.729 323 3080–1.729 323 3082	29	0.172
1.729 323 3083–1.884 615 3846	6	0.167
1.884 615 3847–1.884 615 3959	25	0.16
1.884 615 3960–1.884 617 4277	19	0.158
1.884 617 4278–1.884 983 0076	13	0.154
1.884 983 0077–1.884 984 0227	20	0.15
1.884 984 0228–1.884 984 0255	27	0.148
1.884 984 0256–1.946 428 5714	7	0.143
1.946 428 5715–1.946 428 5720	29	0.138
1.946 428 5721–1.946 428 7922	22	0.136
1.946 428 7923–1.946 508 0621	15	0.133
1.946 508 0622–1.946 508 1722	23	0.130
1.946 508 1723	31	0.129
1.946 508 1724–1.974 137 9310	8	0.125
1.974 137 9311–1.974 137 9568	25	0.12
1.974 137 9569–1.974 156 4838	17	0.118
1.974 156 4839–1.974 156 4967	26	0.115
1.974 156 4968–1.987 288 1355	9	0.111
1.987 288 1356–1.987 288 1387	28	0.107
1.987 288 1388–1.987 292 6211	19	0.105
1.987 292 6212–1.987 292 6226	29	0.103
1.987 292 6227–1.993 697 4789	10	0.1

$2, \ldots\}$, have no common divisor other than one and are not zero at the same time, and $j = 4, 5, 6, \ldots$. In figure 2, we show the main periods j in the $h - l$ plane up to $j = 9$ where l is assumed to be real. Many periods exist in between the main periods but we cannot depict them since the regions for them are so narrow. We give some of them numerically in table 1, where we list the critical values of h, at which the period changes, within ten places of decimals for $l = 3$. The values $(m+n)/[jm+(j+1)n]$ of other periods within ten places of decimals are as follows: $\frac{3}{31}$, $\frac{2}{21}$, $\frac{3}{32}$, $\frac{1}{11}$, $\frac{3}{34}$, $\frac{2}{23}$, $\frac{3}{35}$, $\frac{1}{12}$, $\frac{3}{37}$, $\frac{2}{25}$, $\frac{1}{13}$, $\frac{2}{27}$, $\frac{1}{14}$, $\frac{2}{29}$, $\frac{1}{15}$, $\frac{2}{31}$, $\frac{1}{16}$, $\frac{2}{33}$, $\frac{1}{17}$, $\frac{2}{35}$, $\frac{1}{18}$, $\frac{1}{19}, \ldots, \frac{1}{36}$. Higher periodic points were also found when we took more places of decimals of the value of h. The behaviours of values $(m+n)/[jm+(j+1)n]$ as a function of h, and also as a function of l, show a devil's staircase. Similar behaviours have been found for a number of different systems (Aubrey 1978, Bak 1982, Kaneko 1983).

Properties of the Ising model at the central part of the Cayley tree with $l = 3$ (i.e. coordination number 6) considered here are regarded as those by the Bethe approximation of the Ising model on the triangular lattice in which each lattice site has three bonds of J and three bonds of $-J$ as, for example, shown in figure 3 and we expect that some corresponding properties may appear in that system. In the case of $l = 2$ and also of $l_+ \neq l_-$, the situation is more complicated and we will discuss it elsewhere.

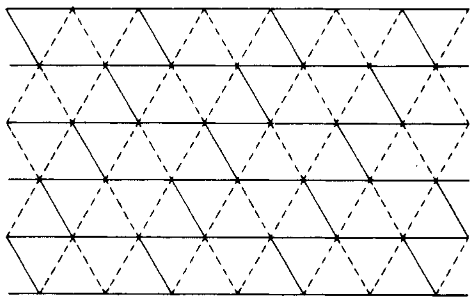

Figure 3. An Ising model on the triangular lattice. The full lines denote ferromagnetic bonds J and the broken lines antiferromagnetic bonds $-J$.

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